PRICING COMMODITY FUTURES CONTRACTS WITH A REGIME-SWITCHING MODEL

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ABSTRACT. In this paper we present one factor model of commodity prices with a single jump regime-switching process. And we derive an analytic formula for pricing futures contracts when the parameters of commoditiy process have governed by a Markov regime-switching process.

1. Introduction

As the development of the financial markets, it is widely used that the derivatives of commodity products in asset portfolio allocation. Especially, the futures contracts of commodity are widely used because of the risk of physical delivery settlement. So the underlying assets of various derivatives of commodity are not spot prices but futures contracts of commodity. Therefore, pricing of futurues of commodity is a more important issue. Since futures contracts are traded on several exchanges, futures contracts have high liquidity and futures prices are more easily observed than the spot prices. The futures prices are significant rather than spot prices in the markets.

We focus on a one-factor commodity model which is suggested by Schwartz([6]). And we apply a single regime-switching model to a one-factor commodity model. In general, a regime-switching model consists of several regimes; within each regime the commodity prices follow a distinct stochastic process. The price process can randomly shift between these regimes. Zhang and Zhou([7]) use a regime-switching model which has a single jump. This is the simplest stochastic volatility model. And they value the stock loans in which the stock price is governed by geometric Brownian motion with a regime switching.

Since a regime-switching model has the empirical importance in financial markets, this model is applied to various derivatives pricing models. Fuh *et al.*([3]) provided a closed-form formula for the value of an European call option and Roh([5]) valued a variance swap when a volatility of underlying assett is modeled as a Markov regime-swtiching process with a finite state.

Chen and Forsyth([1]) have shown that a one-factor regime-switching model outperforms a typical one-factor mean-reverting model in terms of capturing the dynamics of the gas forward curves. Also Choi and Hammoudeh([2]) invest shifts in the volatility between two regimes of commodity prices and US stock markets in a Markov regime-switching environment.

Hansen and Poulsen([4]) extend the short rate Vasicek model to include jumps in the local mean. This model is similar to a mean reverting process of the Ornstein-Uhlenbeck type. So we develop this model for including jumps in the long term mean and the volatility of the commodity price process.

2. The model

We assume that there is a filtered probablility space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, P)$. The filtration $\{\mathcal{F}\}_{t \geq 0}$ satisfies the usual conditions and is generated by one-dimensional Brownian motion B_t and a Markov chain ϵ_t. And we suppose that a Markov chain ϵ_t is independent of B_t and the spot price process of commodity evolves acoording to the form,

$$(2.1) \qquad dS_t = \kappa(\mu(t) - \ln S_t)S_t dt + \sigma(t)S_t dB_t.$$

This means that the logarithm of the spot price of commodity is assumed to follow a mean reverting process which is suggested by Schwartz([6]).

We define $X_t = \ln S_t$, then the equation (2.1) is converted to

$$dX_t = \kappa(\hat{\alpha}(t) - X_t)dt + \sigma(t)dB_t,$$

where $\hat{\alpha}(t) = \mu(t) - \dfrac{\sigma^2(t)}{2\kappa}$. We assume that the long term mean $\hat{\alpha}(t)$ and the volatility $\sigma(t)$ are affected by a continous-time Markov chain ϵ_t. We assume that there are two regime states, high(H) and low(L) states. The corresponding pairs of long term means and volatilities are denoted by $\{\hat{\alpha}_H, \sigma_H\}$ and $\{\hat{\alpha}_L, \sigma_L\}$, respectively. And regime i switches ino regime j at the first jump time of an independent Poisson process with intensity λ_i, for $i, j \in \{H, L\}$. In other words, the random time τ_i

of the leaving regime i has an exponential distribution with intensity λ_i, i.e $P(\tau_i > t) = e^{-\lambda_i t}, i \in \{H, L\}$.

Following the argument of Hansen and Poulsen([4]) and Schwartz([6]), the dynamics of X_t can be written as

$$(2.2) \qquad dX_t = \kappa(\alpha(t) - X_t)dt + \sigma(t)dB_t^Q,$$

where $\alpha(t) = \hat{\alpha}(t) - \nu$, ν is the market price of risk under an equivalent martingale measure Q. For convenience we regard that there is a complete probability space $(\Omega, \mathcal{F}^Q, \{\mathcal{F}^Q\}_{t \geq 0}, Q)$ under an equivalent martingale measure Q.

The futures price of the commodity with maturity T is the expected value of the commodity at the present time t. If $\alpha(s)$ and $\sigma(s)$ are constants for all $s \in (t, T)$, then the futures price of commodity is obtained by the following Proposition.

PROPOSITION 2.1. *The commodity price process follows the stochastic process, equation (2.2) and $\alpha(t), \sigma(t)$ are constants α, σ, respectively. Then the futures price of the commodity with maturity T is the expected price of the commodity under the risk-neutral measure Q. And we obtain the following futures price,*

$$F(S_0, T) = \exp\left\{e^{-\kappa T}\ln S_0 + (1 - e^{-\kappa T})\alpha + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T})\right\}.$$

It is well-known results, so we omit the proof of Proposition 2.1 (See Schwartz([6]) for more details).

Now we consider that $\alpha(s)$ and $\sigma(s)$ are affected by a Markov chain ϵ_t. Then the random time τ_i means the jumping time from i state to j state. Therefore, we obtain the following form of the commodity price process,

$$(2.3) \quad S_t = \exp\left\{e^{-\kappa t}\left(\ln S_0 + \int_0^t \kappa e^{\kappa s}\alpha(s)ds + \int_0^t e^{\kappa s}\sigma(s)dB_s^Q\right)\right\}.$$

Let a regime-switching model have a single jump over the life time of futures. This is the simplest stochastic volatility model and this is used in Zhang and Zhou([7]). In general, regime shifts are occurred with the long term consequence, whereas the futures contracts which are traded in exchanges have a short term to maturity. So assuming a single jump model is sufficient to describe the behavior of futures contracts.

We suppose that the generator of a Markov chain ϵ_t has the form

$$\begin{pmatrix} -\lambda_i & \lambda_i \\ 0 & 0 \end{pmatrix}, \text{with a positive intensity} \lambda_i, i \in \{H, L\}.$$

Then the j-state is the absorbing state.

The following theorem gives the pricing of commodity futrues contracts with a single jump regime-swtiching model.

THEOREM 2.2. *Let $F_i(S_0, T)$ be the value of the commodity futures in initial state i with maturity T, then it satisfies*

$$F_i(S_0, T) = \exp\left\{ e^{-\kappa T} \ln S_0 + (\alpha_j - e^{-\kappa T}\alpha_i) + \frac{1}{4\kappa}(\sigma_j^2 - e^{-2\kappa T}\sigma_i^2) \right\}$$

$$\times \int_0^T \exp\left\{ e^{-\kappa(T-\tau)}(\alpha_i - \alpha_j) + \frac{e^{-2\kappa(t-\tau)}}{4\kappa}(\sigma_i^2 - \sigma_j^2) - \lambda_i\tau \right\} d\tau.$$

Proof. Since the futures price of the commodity with maturity T is the expected price of the commodity at the initial time under risk-neutral measure Q. And we obtain the following futures price,

(2.4) $$F_i(S_0, T) = E[S_T | \epsilon_0 = i]$$

By the property of conditional expectation and a single jump regime-switching model, the right-hand side of equation (2.4) should be equivalent to $\int_0^T E[S_T | \epsilon_0 = i, \tau_i = \tau] \cdot P(\tau) d\tau$, where $P(\tau)$ is the probability of jumping time from i state to j state. And we substitute S_T in the expectation with the equation (2.3). Then we obtain the following equation,

$$F_i(S_0, T)$$

$$= \int_0^T \exp\left\{ e^{-\kappa T} \ln S_0 \right\} \times E\left[\exp\left\{ e^{-\kappa T}\left(\int_0^\tau \kappa e^{\kappa s}\alpha_i ds \right.\right.\right.$$

$$\left.\left.\left. + \int_\tau^T \kappa e^{\kappa s}\alpha_j ds + \int_0^\tau e^{\kappa s}\sigma_i dB_s + \int_\tau^T e^{\kappa s}\sigma_j dB_s \right) \right\} \right] e^{-\lambda_i\tau} d\tau$$

$$= \exp\left\{ e^{-\kappa T} \ln S_0 \right\} \int_0^T \exp\left\{ e^{-\kappa T}(e^{\kappa\tau} - 1)\alpha_i + (e^{\kappa T} - e^{\kappa\tau})\alpha_j) \right.$$

$$\left. + e^{-2\kappa T}\left(\frac{\sigma_i^2(e^{2\kappa\tau} - 1)}{4\kappa} + \frac{\sigma_j^2(e^{2\kappa T} - e^{-2\kappa\tau})}{4\kappa} \right) \right\} \times e^{-\lambda_i\tau} d\tau.$$

Hence the value of the commodity futrures $F_i(S_0, T)$ is

$$\exp\left\{ e^{-\kappa T} \ln S_0 + (\alpha_j - e^{-\kappa T}\alpha_i) + \frac{1}{4\kappa}(\sigma_j^2 - e^{-2\kappa T}\sigma_i^2) \right\}$$

$$\times \int_0^T \exp\left\{ e^{-\kappa(T-\tau)}(\alpha_i - \alpha_j) + \frac{e^{-2\kappa(t-\tau)}}{4\kappa}(\sigma_i^2 - \sigma_j^2) - \lambda_i\tau \right\} d\tau.$$

$\square$

3. Conclusion

In this paper, we have valued a commodity futures constract when the long term mean and the volatility of the spot price process of commodity are modeled as a single jump regime switching process with two states. These results can be used as an analytic formula to derivatives of commodity when the parameters of commodity process have a single jump on the life of a contracts.

References

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